

PROOF OF THE INDEX CONJECTURE IN HOFER GEOMETRY

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ABSTRACT. Let γ be an S^1 -subgroup in $Ham(M, \omega)$ generated by a Morse Hamiltonian H . We give a simple proof of the conjecture stated in [7], relating the Morse index of γ , as a critical point of the Hofer length functional, with the Conley Zehnder index of the extremizers of H , considered as periodic orbits.

1. INTRODUCTION

There has not been much study of the Morse index of geodesics for the Hofer length functional on path spaces of the group of Hamiltonian diffeomorphisms $Ham(M, \omega)$. Maybe this is because the problem of Morse theory for the Hofer length functional seems completely hopeless. This is possibly true to a large extent, however in [7] we showed that doing Morse theory for the Hofer length functional “virtually” can give some interesting results in symplectic topology.

In the special case where γ is an S^1 -subgroup in $Ham(M, \omega)$ generated by a Morse Hamiltonian H , a key point in [7] was using a relationship of the Morse index of γ with the Conley-Zehnder index of the linearized flow at the extremizers of H , in some special cases.

Remark 1.1. *We didn't use the words Conley-Zehnder index in [7], but rather the index of a certain Cauchy-Riemann operator, but this could be directly related to the above CZ index.*

Indeed as a byproduct we arrived at the conjecture that the two indexes must coincide. An inequality relating the two quantities was proved by Karshon-Slimowitz in [2] by constructing a beautiful explicit local family of shortenings of γ .

Here we give a simple proof of the conjecture using calculus of variations, already worked out in [8] for the Hofer length functional. Although the proof is an elementary application of work in [8], we feel that the result is still unexpected. It would also be very surprising if this generalized to arbitrary geodesics in $Ham(M, \omega)$. Of course what may be more interesting than the identity itself is the fact that it is predicted by Gromov-Witten theory in [7]; we feel that there must be more to this story that is left to be explored and so the above coincidence should be made explicit.

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2.1. The group of Hamiltonian symplectomorphisms and Hofer metric.

Given a smooth function $H : M^{2n} \times [0, 1] \rightarrow \mathbb{R}$, there is an associated time dependent Hamiltonian vector field X_t , $0 \leq t \leq 1$, defined by

$$(2.1) \quad \omega(X_t, \cdot) = -dH_t(\cdot).$$

The vector field X_t generates a path $\gamma : [0, 1] \rightarrow \text{Diff}(M)$. Given such a path γ , its end point $\gamma(1)$ is called a Hamiltonian symplectomorphism. The space of Hamiltonian symplectomorphisms forms a group, denoted by $\text{Ham}(M, \omega)$.

In particular the path γ above lies in $\text{Ham}(M, \omega)$. It is well-known that any smooth path γ in $\text{Ham}(M, \omega)$ with $\gamma(0) = \text{id}$ arises in this way (is generated by $H : M \rightarrow \mathbb{R}$ as above). Given such a path γ , the *Hofer length*, $L(\gamma)$ is defined by

$$L(\gamma) := \int_0^1 \max_M H_t^\gamma - \min_M H_t^\gamma dt,$$

where H^γ is a generating function for the path $t \mapsto \gamma(0)^{-1}\gamma(t)$, $0 \leq t \leq 1$. The Hofer distance $\rho(\phi, \psi)$ is defined by taking the infimum of the Hofer length of paths from ϕ to ψ . We only mention it, to emphasize that it is a deep and interesting theorem that the resulting metric is non-degenerate, (cf. [1, 3]). This gives $\text{Ham}(M, \omega)$ the structure of a Finsler manifold.

We now consider L as a functional on the space of paths in $\text{Ham}(M, \omega)$ with fixed end points. For an S^1 subgroup $\gamma : S^1 \rightarrow \text{Ham}(M, \omega)$, generated by a Morse Hamiltonian H , it is shown by Ustilovsky that γ is a smooth critical point of L . Consequently it makes sense to ask for its Morse index, (which might be infinite.) Moreover, it is easy to see that $\text{index}_L(\gamma) = \text{index}_{L_+}(\gamma) + \text{index}_{L_-}(\gamma)$, where:

$$(2.2) \quad L_+(\gamma) := \int_0^1 \max(H_t^\gamma) dt,$$

for H_t^γ in addition normalized by the condition:

$$(2.3) \quad \int_M H_t^\gamma \cdot \omega^n = 0.$$

And where L_- is similarly defined. It will be the Morse index of γ with respect to L_+ that we compute.

We normalize the Conley Zehnder index CZ by the condition that for a C^2 -small function f , the Conley-Zehnder index of a critical point of f (considered as a time 1, periodic orbit of the Hamiltonian flow of f) is the Morse index of the critical point.

For γ as above by the Atiyah-Guillemin-Sternberg convexity theorem there is a unique maximizer x_{\max} of H on M . We will denote the associated periodic orbit by x_{\max} .

Theorem 2.1. *For γ as above, the Morse index of γ with respect to L_+ is*

$$CZ(x_{\max}) - 2n.$$

Proof. The Morse index theorem [4] cannot be directly applied to

$$L_+ = \int_0^1 L(\dot{\gamma}(t), \gamma(t)) dt,$$

$L(\dot{\gamma}(t), \gamma(t)) = \max_M H_t$, for $H_t = \dot{\gamma}(t) \in T_{\gamma(t)}\text{Ham}(M, \omega) \equiv C_{\text{norm}}^\infty(M)$, with the latter being smooth functions normalized to have zero mean, (2.3). This is because

it clearly does not satisfy the Legendre condition that $\frac{d^2}{d\xi^2}L(\dot{\gamma}(t), \gamma(t)) > 0$, for every variation ξ of $\dot{\gamma}$, (for every t).

However Ustilovsky shows that there is a related functional l_+ on a related space, with a critical point which we also denote by γ , s.t. the Hessian of l_+ at γ coincides with the Hessian of L_+ at γ , and to which Morse theorem does apply. This is beautiful, but we refer the reader to [8] and [5] for further details.

Morse theorem gives us the following procedure for the calculation of the Morse index of γ with respect to l_+ . Denote by γ_τ the restriction of γ to $[0, \tau] \subset [0, 1]$. Then $\text{index}(\gamma_\tau)$ is a locally constant, lower semi-continuous function in τ , and jumps at a discrete set of $\tau_i \in [0, 1]$ called conjugate points, (this is our shorthand, usually $\gamma(\tau_i)$ are called points to conjugate to $\gamma(0)$ along γ). The value of the jump $\text{mult}(\tau_i)$ is the dimension of the solution space of the associated Jacobi equation. Informally speaking this is dimension of the space of infinitesimal variations of γ_τ through extremals with the same endpoints. And a point $\tau \in [0, T]$ is defined to be a conjugate point if this dimension is non zero.

In the case of the functional l_+ , it is shown in [8] that $\tau_0 \in [0, 1]$ is a conjugate point if and only if the time τ_0 linearized flow of H at the extremizer x_{\max} of H has periodic orbits, and the multiplicity $\text{mult}(\tau_0)$ is the dimension of the space of these periodic orbits.

The linearized flow of H at x_{\max} , is conjugate to an action of S^1 on \mathbb{C}^n , which must split into invariant 1-complex dimensional subspaces N_{k_i} , on which γ is acting by

$$(2.4) \quad v \mapsto e^{2\pi i k_i t} v.$$

These k_i are defined to be the weights of the circle action γ . Our conventions are

$$\begin{aligned} \omega(X_H, \cdot) &= -dH(\cdot) \\ \omega(\cdot, J\cdot) &> 0. \end{aligned}$$

With these conventions the above weights are all negative, since the Hessian of H at x_{\max} has negative eigenvalues. It is then clear that the Morse index of our γ is given by

$$\sum_{1 \leq i \leq n} 2(|k_i| - 1),$$

the -1 is due to lower semi-continuity of Morse index and conjugacy of the end points for γ . But this is $CZ(x_{\max}) - 2n$, as $CZ(x_{\max})$ is by definition (and our normalization) $2n$ plus the Maslov index for the path of linear Lagrangian subspaces of R^{2n} induced by the linearized flow of H at x_{\max} , see [6]. This Maslov index reduces by axioms, from the decomposition (2.4) to (2.1), after appropriate extension of Maslov index to paths with endpoints not in general position, to account for -1 in the expression above. Or to state this in another way, we have that the Morse index of the geodesic $\tilde{\gamma} = \gamma|_{[\epsilon, 1-\epsilon]}$ is $CZ(x_{\max}) - 2n$, for a small $\epsilon > 0$ where x_{\max} now denotes the constant periodic orbit of the isotopy given by $\tilde{\gamma}$. \square

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